

On Induced Subgraphs of Finite Graphs not Containing Large Empty and Complete Subgraphs

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Abstract

In their celebrated paper [5], Erdős and Hajnal asked the following: is it true, that for any finite graph \mathcal{H} there exists a constant $c(\mathcal{H})$ such that for any finite graph \mathcal{G} , if \mathcal{G} does not contain complete or empty induced subgraphs of size at least $|V(\mathcal{G})|^{c(\mathcal{H})}$, then \mathcal{H} can be isomorphically embedded into \mathcal{G} ? The positive answer has become known as the Erdős-Hajnal conjecture.

In Theorem 3.13 of the present paper we settle this conjecture in the affirmative. To do so, we are studying here the fine structure of ultraproducts of finite sets, so our investigations have a model theoretic character.

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1 Introduction

In their celebrated paper [5], Erdős and Hajnal asked the following question: is it true, that for any finite graph \mathcal{H} there exists a constant $c(\mathcal{H})$ such that for any finite graph \mathcal{G} , if \mathcal{G} does not contain complete or empty induced subgraphs of size at least $|V(\mathcal{G})|^{c(\mathcal{H})}$, then \mathcal{H} can be isomorphically embedded into \mathcal{G} ? The positive answer has become known as the Erdős-Hajnal conjecture.

It is known to be true for a few special cases. In [5] Erdős and Hajnal proved a somewhat weaker general result: for any finite graph \mathcal{H} there exists a constant $c(\mathcal{H})$ such that for any finite graph \mathcal{G} , if \mathcal{G} does not contain complete or empty induced subgraphs of size at least $e^{c(\mathcal{H})} \sqrt{\log_2 |V(\mathcal{G})|}$, then \mathcal{H} can be isomorphically embedded into \mathcal{G} . It is also known from [6], that if \mathcal{G} does not contain complete, or empty

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induced bipartite subgraphs with both parts of size polynomial in $|V(\mathcal{G})|$, then \mathcal{H} can be embedded into \mathcal{G} . In [3] it was shown, that for $k \geq 4$, the class of k -uniform hypergraphs does not satisfy the natural generalization of the Erdős-Hajnal Conjecture. Further interesting related results can be found in [8]. For a comprehensive survey we refer to [2] and the references therein.

In Theorem 3.13 of the present paper we settle the original conjecture in the affirmative. The rest of the paper is devoted to work out the technical details we need to prove our main theorem 3.13. We are studying here the fine structure of ultraproducts of finite sets, consequently, our investigations have a model theoretic character. Our methods have been inspired by nonstandard measure theory; however, our presentation does not refer to any purely measure theoretic method or result. We note, that utilizing a more or less same method, several results (in rather different areas) have been obtained recently. In this respect we refer to [14] [13], [17] and [11].

The rest of the paper is organized as follows. In Section 2 we are recalling the notions and methods we will need later. This section has a survey character in order to keep the present paper self contained. In this section there are no new results. In Section 3 we present all the proofs we need to derive Theorem 3.13 which is the main result of the present paper. As we mentioned, Theorem 3.13 settles the Erdős-Hajnal conjecture affirmatively. Finally, in Section 4 we present some problems which remained open. Some of these problems can be regarded as certain generalizations of the Erdős-Hajnal conjecture, some other problems are related to the methods and results obtained in Section 3.

2 Preliminaries

We start by summing up our system of notation, which is mostly standard. After this, we recall the notions and results we need to establish the proof of our main theorem.

Throughout ω denotes the set of natural numbers and for every $n \in \omega$ we have $n = \{0, 1, \dots, n-1\}$. Let A and B be sets. Then AB denotes the set of functions whose domain is A and whose range is a subset of B . Moreover, ${}^{<\omega}A$ is defined to be ${}^{<\omega}A = \cup_{n \in \omega} {}^nA$. In addition, $|A|$ denotes the cardinality of A ; if κ is a cardinal then $[A]^\kappa$ denotes the set of subsets of A which are of cardinality κ and $\mathcal{P}(A)$ denotes the power set of A , that is, $\mathcal{P}(A)$ consists of all subsets of A .

\mathbb{R} denotes the set of real numbers; \mathbb{R}^+ denotes the set of positive real numbers. If $c \in \mathbb{R}^+$ then we follow the convention that $c/0 = \infty$. Throughout we use function composition in such a way that the rightmost factor acts first. That is, for functions f, g we define $f \circ g(x) = f(g(x))$.

If I is a set, \mathcal{A}_i is a structure for all $i \in I$ and $\mathcal{F} \subseteq \mathcal{P}(I)$ is an ultrafilter, then $\prod_{i \in I} \mathcal{A}_i / \mathcal{F}$ denotes the ultraproduct of the \mathcal{A}_i modulo \mathcal{F} .

Graphs. By a graph we mean a simple, undirected graph $\mathcal{G} = \langle V(\mathcal{G}), E(\mathcal{G}) \rangle$, where $V(\mathcal{G})$ is the set of vertices of \mathcal{G} (which may be finite or infinite) and $E(\mathcal{G})$ is the set of edges of \mathcal{G} . Let $X \subseteq V(\mathcal{G})$. Then $\Gamma(X)$ denotes the set of vertices of \mathcal{G} connected to some elements of X :

$$\Gamma(X) = \{a \in V(\mathcal{G}) : (\exists b \in X)(\langle a, b \rangle \in E(\mathcal{G}))\}.$$

If $a \in V(\mathcal{G})$ then instead of $\Gamma(\{a\})$ we will simply write $\Gamma(a)$.

Ultratopologies. Now we recall some parts of [12] and [9]; for further related results we also refer to [15].

Definition 2.1 *Let $\langle C_i : i \in I \rangle$ be a sequence of arbitrary sets, let $k \in \omega$ and let \mathcal{F} be an ultrafilter over I . A k -ary relation $X \subseteq {}^k(\Pi_{i \in I} C_i / \mathcal{F})$ is defined to be decomposable iff for each $i \in I$, there are $X_i \subseteq {}^k C_i$ such that $X = \Pi_{i \in I} X_i / \mathcal{F}$.*

Decomposable relations may be characterized in terms of some topologies defined naturally on ultraproducts.

Definition 2.2 *Let $k \in \omega$. Then a function $\hat{\cdot} : {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F}) \rightarrow {}^k(\Pi_{i \in I} \mathcal{A}_i)$ is defined to be a k -dimensional choice function iff for all $\bar{a} \in {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ we have $\bar{a} = \hat{\bar{a}} / \mathcal{F}$.*

For a given choice function $\hat{\cdot}$, $\bar{a} \in {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ and $X \subseteq {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ define $T(\bar{a}, X)$ as follows:

$$T(\bar{a}, X) = \{i \in I : (\exists \bar{b} \in X)(\hat{\bar{a}}(i) = \hat{\bar{b}}(i))\}.$$

A set $X \subseteq {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ is defined to be closed with respect to $\hat{\cdot}$, iff for all $\bar{a} \in {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ we have

$$T(\bar{a}, X) \in \mathcal{F} \text{ implies } \bar{a} \in X.$$

Let $\hat{\cdot}$ be a fixed k -dimensional choice function. For completeness, we note, that the family of k -ary relations $X \subseteq {}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$ closed with respect to $\hat{\cdot}$, is the family of all closed sets of a topological space on ${}^k(\Pi_{i \in I} \mathcal{A}_i / \mathcal{F})$. These spaces are called *ultratopologies*. For the details we refer to [12] and [9].

Lower Cofinalities of Ultrafilters. Suppose $\mathcal{F} \subseteq \mathcal{P}(I)$ is an ultrafilter and let κ be a regular cardinal. The natural order of κ is the ordering relation of ordinals restricted to κ ; it will be denoted by $<_\kappa$. An element $a \in {}^I \kappa / \mathcal{F}$ is defined to be unbounded iff for all $n \in \kappa$ we have $\{i \in I : \hat{a}(i) \geq n\} \in \mathcal{F}$. Recall Definition VI.3.5 from [16]: the lower cofinality $lc f(\kappa, \mathcal{F})$ is the smallest cardinality λ such that there exists a λ -sized set of unbounded elements of ${}^I \kappa / \mathcal{F}$ which is unbounded from below (according to the ultraproduct-order ${}^I <_\kappa / \mathcal{F}$ of the natural order $<_\kappa$ of κ).

3 Proofs

In this section we present the lemmas and their proofs we need to establish our main result: Theorem 3.13.

Throughout I is a set, A_i is a finite set, $\mathcal{G}_i = \langle A_i, E_i \rangle$ is a simple, undirected graph for all $i \in I$ and $\mathcal{F} \subseteq \mathcal{P}(I)$ is a nonprincipal ultrafilter on I . In addition, $A = \prod_{i \in I} A_i / \mathcal{F}$, $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i / \mathcal{F}$. To denote the vertex set of \mathcal{G}_i , in place of A_i , we may write $V(\mathcal{G}_i)$, as well.

Definition 3.1 *Let B be a set. A function $X : {}^{<\omega}2 \rightarrow \mathcal{P}(B)$ is defined to be a Hausdorff scheme over B iff the following stipulations are satisfied for all $p \in {}^{<\omega}2$:*

$$X_{p \smallfrown 0} \cap X_{p \smallfrown 1} = \emptyset;$$

$$X_{p \smallfrown 0}, X_{p \smallfrown 1} \subseteq X_p.$$

Thus, a Hausdorff scheme is an infinite binary tree from the subsets of B . Above, and thereafter we write X_p instead of $X(p)$.

A Hausdorff scheme over A is defined to be decomposable iff X_p is a decomposable subset of A for any $p \in {}^{<\omega}2$.

Definition 3.2 *Let $X = \langle X_i : i \in I \rangle / \mathcal{F}$ and $Y = \langle Y_i : i \in I \rangle / \mathcal{F}$ be decomposable subsets of A . X is defined to be big iff there exists $c \in \mathfrak{R}^+$ such that*

$$(*) \quad \{i \in I : |X_i| \geq |A_i|^c\} \in \mathcal{F}.$$

Note, that although a decomposable $X \subseteq A$ may have different decompositions, the truth of $()$ does not depend on the particular choice of $\langle X_i : i \in I \rangle / \mathcal{F}$.*

In addition, X is defined to be small iff it is not big.

Lemma 3.3 *For any $i \in I$ let $\langle B_j^i : j \in \omega \rangle$ be a decreasing sequence of subsets of A_i . If $f, g : I \rightarrow \omega$ are such that $K := \{i \in I : f(i) \geq g(i)\} \in \mathcal{F}$ then*

$$\prod_{i \in I} B_{f(i)}^i / \mathcal{F} \subseteq \prod_{i \in I} B_{g(i)}^i / \mathcal{F}.$$

Proof. Let $s = \langle s_i : i \in I \rangle / \mathcal{F} \in \prod_{i \in I} B_{f(i)}^i / \mathcal{F}$ be arbitrary. Let $J = \{i \in I : s_i \in B_{f(i)}^i\}$. Clearly, $J \cap K \in \mathcal{F}$. In addition, for any $i \in J \cap K$ we have $s_i \in B_{f(i)}^i \subseteq B_{g(i)}^i$. Hence $\{i \in I : s_i \in B_{g(i)}^i\} \supseteq J \cap K$, whence the statement follows. ■

Definition 3.4 *Let $X = \langle X_p : p \in {}^{<\omega}2 \rangle$ be a decomposed Hausdorff scheme of A such that $X_p = \langle (X_p)_i : i \in I \rangle / \mathcal{F}$ for all $p \in {}^{<\omega}2$. Then*

$$B_j^i(X) = \bigcup_{p \in {}^j 2} (X_p)_i.$$

When X is clear from the context, we may simply write B_j^i in place of $B_j^i(X)$.

Let

$$S := \bigcup_{f \in {}^\omega 2} \bigcap_{n \in \omega} X_{f|_n}$$

and let $b : \omega \rightarrow {}^I \omega$ be an unbounded function, that is $\{i \in I : b(i) \geq n\} \in \mathcal{F}$ hold, for all $n \in \omega$. Let $s \in \prod_{i \in I} A_i$ and $f \in {}^\omega 2$ be such that $s/\mathcal{F} \in \bigcap_{n \in \omega} X_{f|_n} \subseteq S$. Then the function $\partial s : \omega \rightarrow \omega$ is defined to be

$$\partial s(i) = \sup\{j \leq b(i) : s_i \in (X_{f|_j})_i\}.$$

Clearly, $\partial s(i)$ is finite (in fact, $\partial s(i) < b(i)$); throughout the whole paper b will be fixed, its only role is to keep the values of the ∂s finite - this is the reason why we do not abbreviate b in ∂s).

Lemma 3.5 *Then there exists a sequence $\langle f_\alpha \in {}^I \omega : \alpha < \text{pcf}(\aleph_0, \mathcal{F}) \rangle$ such that the following hold for all $\alpha, \beta < \text{pcf}(\aleph_0, \mathcal{F})$ and $n \in \omega$*

- (i) f_α is unbounded, that is, $\{i \in I : f_\alpha(i) \geq n\} \in \mathcal{F}$;
- (ii) $\alpha < \beta$ implies $f_\beta <_{\mathcal{F}} f_\alpha$, that is, $\{i \in I : f_\beta(i) < f_\alpha(i)\} \in \mathcal{F}$;
- (iii) for all unbounded $h \in {}^I \omega$ there exists $\alpha < \text{pcf}(\aleph_0, \mathcal{F})$ with $f_\alpha <_{\mathcal{F}} h$.

Proof. Let $\{g_\alpha : \alpha < \text{pcf}(\aleph_0, \mathcal{F})\}$ be a cofinal subset of the set of unbounded elements of ${}^I \aleph_0 / \mathcal{F}$ (ordered by the reverse of the ultrapower of the natural ordering of ω). We define the sequence $\langle f_\alpha \in {}^I \omega : \alpha < \text{pcf}(\aleph_0, \mathcal{F}) \rangle$ by transfinite recursion as follows. Suppose $\gamma < \text{pcf}(\aleph_0, \mathcal{F})$ and for any $\alpha < \gamma$ the function f_α has already been defined such that the following stipulations are satisfied:

- (a) f_α is unbounded;
- (b) $\alpha < \beta < \gamma$ implies $f_\beta <_{\mathcal{F}} f_\alpha$;
- (c) $\alpha < \beta < \gamma$ implies $f_\beta <_{\mathcal{F}} g_\alpha$.

Let $F = \{f_\alpha, g_\alpha : \alpha < \gamma\}$. Since $\gamma < \text{pcf}(\aleph_0, \mathcal{F})$, it follows, that $|F| < \text{pcf}(\aleph_0, \mathcal{F})$, hence, there exists an unbounded $f_\gamma \in {}^I \omega$ which is a lower bound of the set F . Clearly, (a)-(c) remain true. In this way the sequence $\langle f_\alpha \in {}^I \omega : \alpha < \text{pcf}(\aleph_0, \mathcal{F}) \rangle$ can be completely built up. Clearly, (a) implies (i) and (b) implies (ii). Since $\{g_\alpha : \alpha < \text{pcf}(\aleph_0, \mathcal{F})\}$ is cofinal, (c) implies (iii). \blacksquare

Lemma 3.6 *Let $X = \langle X_p : p \in {}^{<\omega} 2 \rangle$ be a decomposable Hausdorff scheme of A . Then*

$$S := \bigcup_{f \in {}^\omega 2} \bigcap_{n \in \omega} X_{f|_n}$$

is a union of an increasing sequence of decomposable subsets of A ; in fact if $\lambda := \text{pcf}(\aleph_0, \mathcal{F})$ then there exists a sequence of unbounded functions $\langle f_\alpha \in {}^I \omega : \alpha < \lambda \rangle$ such that for all $\alpha < \beta < \lambda$ we have $\{i \in I : f_\alpha(i) \geq f_\beta(i)\} \in \mathcal{F}$ and

$$S = \bigcup_{\alpha < \lambda} \prod_{i \in I} B_{f_\alpha(i)}^i / \mathcal{F}.$$

Proof. For each $p \in {}^{<\omega}2$ fix a decomposition $X_p = \langle (X_p)_i : i \in I \rangle / \mathcal{F}$. We may assume

$$(X_{p \smallfrown 0})_i \cap (X_{p \smallfrown 1})_i = \emptyset \text{ and } \\ (X_{p \smallfrown 0})_i, (X_{p \smallfrown 1})_i \subseteq (X_p)_i$$

for every $p \in {}^{<\omega}2$ and $i \in I$. For any $i \in I$ and $j \in \omega$ let $B_j^i = B_j^i(X)$. Clearly, $\langle B_j^i : j \in \omega \rangle$ is a decreasing sequence for any fixed $i \in I$.

Because of the definition of S , for any $s \in S$ there exists (a unique) $g_s \in {}^\omega 2$ such that

$$s \in \bigcap_{n \in \omega} X_{(g_s)|_n}.$$

Let $\langle f_\alpha \in {}^I \omega : \alpha < \lambda \rangle$ be a sequence satisfying the conclusion of Lemma 3.5. Now for any $\alpha < \lambda$ let $D^\alpha = \prod_{i \in I} B_{f_\alpha(i)}^i / \mathcal{F}$. Let $\hat{\cdot}$ be any choice function on A . Observe, that by construction, for all $s \in S$ there exists $\alpha(s) < \lambda$ with $f_{\alpha(s)} <_{\mathcal{F}} \partial \hat{s}$. Hence we have

$$s \in \prod_{i \in I} (X_{(g_s)|_{\partial \hat{s}(i)}})_i / \mathcal{F} \subseteq \prod_{i \in I} (X_{(g_s)|_{f_{\alpha(s)}(i)}})_i / \mathcal{F} \subseteq \prod_{i \in I} B_{f_{\alpha(s)}(i)}^i / \mathcal{F} = D^\alpha.$$

Consequently, $S \subseteq \bigcup_{\alpha < \lambda} D^\alpha$. Hence, to complete the proof, it remains to show the other inclusion: $D^\alpha \subseteq S$ for any $\alpha < \lambda$. Let $\alpha < \lambda$ be fixed. Let $a \in D^\alpha$ be arbitrary. We show, that there is a unique $g_a \in {}^\omega 2$ such that

$$(**) \quad a \in \bigcap_{n \in \omega} X_{(g_a)|_n}.$$

Observe, that uniqueness of g_a follows from the fact, that for all $p \in {}^{<\omega}2$ the sets $X_{p \smallfrown 0}$ and $X_{p \smallfrown 1}$ are disjoint.

To show (**), observe, that for all $i \in I$ there exists a unique $t_{a,i} \in {}^{f_\alpha(i)}2$ such that

$$\{i \in I : \hat{a}_i \in (X_{t_{a,i}})_i\} \in \mathcal{F}.$$

In addition, for all $n \in \omega$ there exists a unique $g_{a,n} \in {}^n 2$ such that

$$(***) \quad \{m \in I : g_{a,n} \subseteq t_{a,m}\} \in \mathcal{F}.$$

It is easy to see, that $n \leq k$ implies $g_{a,n} \subseteq g_{a,k}$. Hence $g_a := \bigcup_{n \in \omega} g_{a,n}$ is a function: $g_a \in {}^\omega 2$. In addition, for all $n \in \omega$, by (***) we have $\{m \in I : \hat{a}_m \in (X_{g_{a,n}})_m\} \in \mathcal{F}$; consequently $a \in X_{(g_a)|_n}$. This completes the proof of (**).

Turning back to show $D^\alpha \subseteq S$ observe

$$D^\alpha \subseteq \bigcup_{g_a : a \in D^\alpha} \bigcap_{n \in \omega} X_{(g_a)|_n} \subseteq S.$$

This completes the proof. ■

Lemma 3.7 *Let $\lambda := \text{pcf}(\aleph_0, \mathcal{F})$ and suppose*

- (i) *\mathcal{F} is λ -regular, that is, there exists $E \in [\mathcal{F}]^\lambda$ such that for all $i \in I$ we have $\nu(i) := \{e \in E : i \in e\}$ is finite;*
- (ii) *for all $i \in I$ we have $2^{|\nu(i)|} \leq |A_i|$.*

Let $X = \langle X_p : p \in {}^{<\omega}2 \rangle$ be a decomposable Hausdorff scheme of A such that for all $p \in {}^{<\omega}2$ the set X_p is big, but for all $i \in I$ we have $|(X_p)_i| \leq |A_i|^{1/|p|}$. Then the following hold.

- (a) *If $f \in {}^\omega 2$ and $B \subseteq A$ is decomposable such that $\bigcap_{n \in \omega} X_{f|_n} \subseteq B$ then B is big;*
- (b) *for any $i \in I$ there exist $T_i \subseteq A_i$ and $l_i \in \omega$ such that for any $a \neq b \in T_i$ there exist $p \neq q \in {}^{l_i}2$ with $a \in (X_p)_i, b \in (X_q)_i$; in addition, $\prod_{i \in I} T_i / \mathcal{F}$ is big.*

Proof. First we show (a). Assume that $B \subseteq A$ is decomposable and small. Then, for any $n \in \omega$ there exists $s \in X_{f|_n} - B$, because $X_{f|_n}$ is big. It follows, that $\{(X_{f|_n}(v) \wedge \neg B(v)) : n \in \omega\}$ is finitely satisfiable (if we treat these conditions as first order formulas in an appropriate language). Consequently, by compactness, there exists $s \in \bigcap_{n \in \omega} X_{f|_n} - B$, so B cannot cover $\bigcap_{n \in \omega} X_{f|_n}$. Hence, if B is decomposable and $\bigcap_{n \in \omega} X_{f|_n} \subseteq B$, then B is big, as desired.

Now we turn to show (b). Let

$$S := \bigcup_{f \in {}^\omega 2} \bigcap_{n \in \omega} X_{f|_n}.$$

By Lemma 3.6 there exists a decreasing sequence of unbounded functions $\langle f_\alpha \in {}^I \omega : \alpha < \lambda \rangle$ such that $S = \bigcup_{\alpha < \lambda} \prod_{i \in I} B_{f_\alpha(i)}^i / \mathcal{F}$. Enumerate E as $E = \{e_\alpha : \alpha < \lambda\}$. We need the following claim.

Claim 3.7.1 *Suppose $e'_\alpha \subseteq e_\alpha$, $e'_\alpha \in \mathcal{F}$ for all $\alpha < \lambda$. Let $\nu'(i) = \{e'_\alpha : i \in e'_\alpha\}$, let $C'_i = \bigcup \{B_{f_\alpha(i)}^i : e'_\alpha \in \nu'(i)\}$ and let $C' = \prod_{i \in I} C'_i / \mathcal{F}$. Then C' is big.*

Proof of Claim 3.7.1. Observe, that for all $\alpha < \lambda$ and $i \in e'_\alpha$ we have that $e'_\alpha \in \nu'(i)$, so $B_{f_\alpha(i)}^i \subseteq C'_i$ holds for all $i \in e'_\alpha$. Hence, for all $\alpha < \lambda$ we have $\prod_{i \in I} B_{f_\alpha(i)}^i / \mathcal{F} \subseteq C'$. It follows, that $S \subseteq C'$. Let $f \in {}^\omega 2$ be arbitrary. Then

$$\bigcap_{n \in \omega} X_{f|_n} \subseteq S \subseteq C'.$$

Thus, by (i) of the present Lemma, C' is big. ■

Next, we show, that there is an $\alpha < \lambda$ such that $D^\alpha := \prod_{i \in I} B_{f_\alpha(i)}^i / \mathcal{F}$ is big. Assume, seeking a contradiction, that D^α is small, for all $\alpha < \lambda$. Let $c \in \mathbb{R}^+$ be arbitrary and fixed. For all $\alpha < \lambda$ let $e'_\alpha = \{i \in e_\alpha : |B_{f_\alpha(i)}^i| < |A_i|^{c/3}\}$. Let $J = \{i \in I : |\nu(i)| < |A_i|^{c/3}\}$. Clearly, $J \in \mathcal{F}$. For all $i \in I$, let $\nu'(i), C'_i$ and C' be as defined in Claim 3.7.1. On one hand, by Claim 3.7.1

$$(*) \quad C' \text{ is big.}$$

On the other hand, for any $i \in J$ we have

$$|C'_i| \leq |\cup \{B_{f_\alpha(i)}^i : e'_\alpha \in \nu'(i)\}| \leq |\nu'(i)| |A_i|^{c/3} \leq |\nu(i)| |A_i|^{c/3} \leq |A_i|^{2c/3}.$$

Since $c \in \mathfrak{R}^+$ was arbitrary, the above computation shows, that C' is small, which contradicts to $(*)$. So there exists $\alpha < \lambda$ such that D^α is big.

For all $i \in I$ let $l_i = f_\alpha(i)$, let $P_i = \{p \in {}^{f_\alpha(i)}2 : (X_p)_i \neq \emptyset\}$ and let $T_i \subseteq B_{f_\alpha(i)}^i \subseteq A_i$ be any set such that for all $p \in P_i$ we have $|T_i \cap (X_p)_i| = 1$. To complete the proof, it is enough to show, that $T := \prod_{i \in I} T_i / \mathcal{F}$ is big. Assume, seeking a contradiction, that T is small. Let $c \in \mathfrak{R}^+$ be arbitrary. Then $J_0 := \{i \in I : |T_i| \leq |A_i|^{c/3}\} \in \mathcal{F}$. Let $n \in \omega$ be such that $1/n \leq c/3$. Since f_α is unbounded, it follows, that $J_1 := \{i \in I : f_\alpha(i) \geq n\} \in \mathcal{F}$. Now, for any $i \in J_0 \cap J_1$ we have

$$|B_{f_\alpha(i)}^i| \leq |T_i| |A_i|^{1/f_\alpha(i)} \leq |A_i|^{c/3} |A_i|^{1/n} \leq |A_i|^{2c/3}.$$

Since $c \in \mathfrak{R}^+$ was arbitrary, it follows, that D^α is small, contradicting to the last line of the previous paragraph. This contradiction completes the proof (it is easy to check, that the T_i satisfy all the other requirements in the statement of the present lemma). \blacksquare

Remark 3.8 Let $f \in {}^\omega 2$ be arbitrary. Keeping the notation introduced in the proof of Lemma 3.7, we note, that $Z := \bigcap_{n \in \omega} X_{f|_n}$ cannot be covered by any small decomposable set, but, Z can be covered by “arbitrarily small big sets”. More precisely, let $c \in \mathfrak{R}^+$, be arbitrary. Choose $n \in \omega$ with $1/n < c$. Then clearly, $X_{f|_n}$ covers Z and

$$\{i \in I : |(X_{f|_n})_i| \leq |A_i|^c\} \in \mathcal{F}.$$

Lemma 3.9 Suppose \mathcal{F} satisfies conditions (i) and (ii) of Lemma 3.7. Suppose, that if $X \subseteq V(\mathcal{G})$ is decomposable and $\mathcal{G}|_X$ is either a complete or empty graph, then X is small.

- (1) If $X \subseteq V(\mathcal{G})$ is big then there exists a big $X' \subseteq X$ such that if $A, B \subseteq X'$ are disjoint and big then there are $a \in A, b \in B$ such that $\langle a, b \rangle \in E(\mathcal{G})$.
- (2) If $X \subseteq V(\mathcal{G})$ is big then there exists a big $X' \subseteq X$ such that if $A, B \subseteq X'$ are disjoint and big then there are $a \in A, b \in B$ such that $\langle a, b \rangle \notin E(\mathcal{G})$.
- (3) If $X \subseteq V(\mathcal{G})$ is big then there exists a big $X' \subseteq X$ such that if $A, B \subseteq X'$ are disjoint and big then there are $a, a' \in A$ and $b, b' \in B$ such that $\langle a, b \rangle \in E(\mathcal{G})$ and $\langle a', b' \rangle \notin E(\mathcal{G})$.

Proof. To prove (1), let $X \subseteq V(\mathcal{G})$ be big and assume seeking a contradiction, that

(*) for any big $X' \subseteq X$ there are $A, B \subseteq X'$ such that A and B are disjoint, big and there are no edges between A and B .

We define a Hausdorff scheme $\langle X_p : p \in {}^{<\omega}2 \rangle$ such that the following stipulations are satisfied for any $p \in {}^{<\omega}2$ and $i \in I$.

- (a) $X_p \subseteq X$ is big (particularly, it is decomposable: $X_p = \langle (X_p)_i : i \in I \rangle / \mathcal{F}$);
- (b) there are no edges between $(X_{p \smallfrown 0})_i$ and $(X_{p \smallfrown 1})_i$, in addition;
- (c) $|(X_p)_i| \leq |A_i|^{1/|p|}$.

Let $X_\emptyset = X$ and assume, that $n \in \omega$ and X_p had already been defined for each $p \in {}^{<n}2$. Let $p \in {}^{n-1}2$ be arbitrary. Combining (a) and (*), it follows, that there are $X_{p \smallfrown 0}, X_{p \smallfrown 1} \subseteq X_p$ such that $X_{p \smallfrown 0}$ and $X_{p \smallfrown 1}$ are disjoint, big and there are no edges between $X_{p \smallfrown 0}$ and $X_{p \smallfrown 1}$. Shrinking their decomposition if necessary, (a)-(c) remains true for them. In this way, $\langle X_p : p \in {}^{<\omega}2 \rangle$ can be completely build up.

According to our stipulations (a) and (c), Lemma 3.7 (b) can be applied: for any $i \in I$ there exist $T_i \subseteq A_i$ and $l_i \in \omega$ such that for any $a \neq b \in T_i$ there exists $p \neq q \in {}^{l_i}2$ with $a \in (X_p)_i, b \in (X_q)_i$; in addition, $T := \prod_{i \in I} T_i / \mathcal{F}$ is big. Combining this with stipulation (b), it follows, that $\mathcal{G}|_T$ is an empty graph. Since T is big, this contradicts to the assumptions of the present lemma.

The proof of (2) is completely analogous with that of (1); as an alternative proof, one may apply (1) directly to the complementer graph of \mathcal{G} .

To show (3), we apply (1) and (2) consecutively: let $X \subseteq A$ be big. Then by (1) there exists a big $Y \subseteq X$ such that if $A, B \subseteq Y$ are disjoint and big, then there exists an edge between some elements of A and B . Now apply (2) to Y to obtain a set X' satisfying the conclusion of (3). \blacksquare

Definition 3.10 Suppose $X_0, \dots, X_{n-1} \subseteq A$ are big sets. Then the point $a \in A$ is defined to be $\langle X_0, \dots, X_{n-1} \rangle$ -generic iff for all $i < n$ the sets $X_i \cap \Gamma(a)$ and $X_i - \Gamma(a)$ are big.

Lemma 3.11 Assume \mathcal{F} satisfies conditions (i) and (ii) of Lemma 3.7. Suppose, that if $X \subseteq V(\mathcal{G})$ is decomposable and $\mathcal{G}|_X$ is either a complete or empty graph, then X is not big. Suppose X' satisfies the conclusion (last two lines) of Lemma 3.9 (3) and suppose $Y_0, \dots, Y_n \subseteq X'$ are disjoint big sets. Then there exists $a \in Y_n$ which is $\langle Y_0, \dots, Y_{n-1} \rangle$ -generic.

Proof. For each $k < n$ fix a decomposition of Y_k , say $Y_k = \langle Y_i(k) : i \in I \rangle / \mathcal{F}$. We define Hausdorff schemes $\langle X(k)_p : p \in {}^{<\omega}2 \rangle$ by recursion such that the following stipulations are satisfied for all $i \in I, p \in {}^{<\omega}2$ and $k < n$:

- (a) $X(k)_p$ is big,
particularly, it is decomposable: $X(k)_p = \langle (X(k)_p)_i : i \in I \rangle / \mathcal{F}$;
- (b) $|(X(k)_p)_i| \leq |A_i|^{1/|p|}$;

$$(c) (X(k)_{p \smallfrown 0})_i \cap (X(k)_{p \smallfrown 1})_i = \emptyset.$$

Let $(X(k)_{\emptyset})_i = Y_i(k)$ and let $X(k)_{\emptyset} = \prod_{i \in I} (X(k)_{\emptyset})_i / \mathcal{F} (= Y_k)$. Now suppose, that $X(k)_p$ has already been defined for some $p \in {}^{<\omega}2$ such that (a)-(c) holds. Then for all $i \in I$ there exist disjoint $(X(k)_{p \smallfrown 0})_i, (X(k)_{p \smallfrown 1})_i \subseteq (X(k)_p)_i$ such that $X(k)_{p \smallfrown 0} := \prod_{i \in I} (X(k)_{p \smallfrown 0})_i / \mathcal{F}$ and $X(k)_{p \smallfrown 1} := \prod_{i \in I} (X(k)_{p \smallfrown 1})_i / \mathcal{F}$ are big. Then (a) and (c) remain true for $X(k)_{p \smallfrown 0}$ and $X(k)_{p \smallfrown 1}$; shrinking $(X(k)_{p \smallfrown 0})_i$ and $(X(k)_{p \smallfrown 1})_i$ if necessary, we may assume, that (b) also remains true. In this way, the Hausdorff schemes $X(k)$ can be completely built up for all $k < n$.

For all $k < n$ and $p \in {}^{<\omega}2$ adjoin to the first order language of graphs a new unary relation symbol, and interpret it in \mathcal{G}_i as $(X(k)_p)_i$. We do not make a strict distinction between these relations and the relation symbols (at the language level) denoting them. In fact, in order to keep notation simpler, we also use $(X(k)_p)_i$ as a relation symbol denoting the relation defined in the previous paragraph. Similarly, by “ Y_n ” we mean a big set, and at the same time, we also use “ Y_n ” as a relation symbol denoting this big set (as a relation). For each $p \in {}^{<\omega}2$ and $k < n$ let $\psi_{p,k}(v)$ be the following first order formula:

$$\psi_{p,k}(v) = (\exists u \in X(k)_p) (\exists w \in X(k)_p) (E(v, u) \wedge \neg E(v, w))$$

and for every $m \in \omega$ let

$$\varphi_m(v) = (v \in Y_n) \wedge \bigwedge_{p \in {}^m 2} \bigwedge_{k < n} \psi_{p,k}(v).$$

Next, we show, that $\varphi_m(v)$ is satisfiable in \mathcal{G} for all $m \in \omega$. To do so, fix $m \in \omega$. For any $p \in {}^m 2$ and $k < n$ let

$$R_{p,k} = \{a \in Y_n : (\forall u \in X(k)_p) (\neg E(a, u))\}$$

and let

$$S_{p,k} = \{a \in Y_n : (\forall w \in X(k)_p) (E(a, w))\}.$$

Assume seeking a contradiction, that

$$Y_n = \bigcup_{p \in {}^m 2, k < n} R_{p,k} \cup S_{p,k}.$$

Observe, that $R_{p,k}$ and $S_{p,k}$ are first order definable from Y_n, E and $X(k)_p$. Hence $R_{p,k}$ and $S_{p,k}$ are decomposable. It follows, that at least one of them is big; by symmetry, we may assume $R_{p,k}$ is big (for a particular p and k). Then, there is no edge between $R_{p,k}$ and $X(k)_p$, which contradicts to Lemma 3.9 (3). Thus, φ_m is satisfiable in \mathcal{G} for all $m \in \omega$.

It is also easy to see, that if $m < j$ then φ_j implies φ_m . It follows, that $\{\varphi_m(v) : m \in \omega\}$ is finitely satisfiable in \mathcal{G} . Hence, by compactness, there exists $a \in Y_n$ simultaneously satisfying $\varphi_m(v)$ for all $m \in \omega$. To complete the proof, it is enough

to show, that $Y_k \cap \Gamma(a)$ and $Y_k - \Gamma(a)$ is big for all $k < n$. Let $k < n$ be fixed. Let $\hat{\cdot}$ be an arbitrary choice function on A . For all $i \in I$ and $p \in {}^{<\omega}2$ let

$$(X'(k)_p)_i = \begin{cases} (X(k)_p)_i & \text{if } \mathcal{G}_i \models \psi_{p,k}(\hat{a}(i)), \\ \emptyset & \text{otherwise.} \end{cases}$$

In addition, let $X'(k)_p = \prod_{i \in I} (X'(k)_p)_i / \mathcal{F}$. Clearly, $X'(k) := \langle X'(k)_p : p \in {}^{<\omega}2 \rangle$ is a Hausdorff scheme. Because of the choice of a , we have $\{i \in I : (X'(k)_p)_i = (X(k)_p)_i\} \in \mathcal{F}$ holds for any $p \in {}^{<\omega}2$. Hence (a)-(c) is true for $X'(k)$, as well, so Lemma 3.7 (b) may be applied to $X'(k)$: for any $i \in I$ there exist $T_i \subseteq A_i$ and $l_i \in \omega$ such that for any $b \neq c \in T_i$ there exists $p \neq q \in {}^{l_i}2$ with $b \in (X'(k)_p)_i$ and $c \in (X'(k)_q)_i$; in addition, $\prod_{i \in I} T_i / \mathcal{F}$ is big.

By construction, if $(X'(k)_p)_i \neq \emptyset$ then there exists $b_{p,i}, c_{p,i} \in (X'(k)_p)_i$ such that $\langle \hat{a}_i, b_{p,i} \rangle \in E(\mathcal{G}_i)$ and $\langle \hat{a}_i, c_{p,i} \rangle \notin E(\mathcal{G}_i)$. Let $H_i = \{b_{p,i} : (X'(k)_p)_i \neq \emptyset\}$ and let $K_i = \{c_{p,i} : (X'(k)_p)_i \neq \emptyset\}$.

One one hand,

$$\prod_{i \in I} H_i / \mathcal{F} \subseteq X'(k)_{\langle \rangle} \cap \Gamma(a) \subseteq Y_k \cap \Gamma(a)$$

and similarly,

$$\prod_{i \in I} K_i / \mathcal{F} \subseteq X'(k)_{\langle \rangle} - \Gamma(a) \subseteq Y_k - \Gamma(a).$$

On the other hand, because of the elements of T_i are separated by the $(X'(k)_p)_i$, we have

$$|\{p \in {}^{l_i}2 : (X'(k)_p)_i \neq \emptyset\}| \geq |T_i|.$$

Consequently $|T_i| \leq |H_i|, |K_i|$ for all $i \in I$, hence $\prod_{i \in I} H_i / \mathcal{F}$ and $\prod_{i \in I} K_i / \mathcal{F}$ are big, as desired. \blacksquare

Theorem 3.12 *Assume \mathcal{F} satisfies conditions (i) and (ii) of Lemma 3.7. Suppose, that if $X \subseteq V(\mathcal{G})$ is decomposable and $\mathcal{G}|_X$ is either a complete or empty graph, then X is not big. Suppose X' satisfies the conclusion (last two lines) of Lemma 3.9 (3) and let $\mathcal{H} = \langle n, E(\mathcal{H}) \rangle$ be a graph on n vertices.*

If $Y_0, \dots, Y_{n-1} \subseteq X'$ are disjoint big sets, then there exists a function $\varrho : n \rightarrow V(\mathcal{G})$ such that for any $i < n$ we have $\varrho(i) \in Y_i$ and ϱ isomorphically embeds \mathcal{H} into \mathcal{G} .

Proof. We apply induction on the number of vertices of \mathcal{H} . If \mathcal{H} has only one vertex, then the statement is trivial. Now assume, that \mathcal{H} has n vertices and the theorem is true for any graph having at most $n - 1$ vertices. Let $Y_0, \dots, Y_{n-1} \subseteq X'$ be disjoint big sets and let $\mathcal{H}' = \mathcal{H}|_{(n-1)}$ be the subgraph of \mathcal{H} induced by its first

$n - 1$ vertices. By Lemma 3.11 there exists $a \in Y_{n-1}$ which is $\langle Y_0, \dots, Y_{n-2} \rangle$ -generic. For any $i < n - 1$ let

$$Y'_i = \begin{cases} Y_i \cap \Gamma(a) & \text{if } \langle i, n - 1 \rangle \in E(\mathcal{H}), \\ Y_i - \Gamma(a) & \text{otherwise.} \end{cases}$$

Since a is $\langle Y_0, \dots, Y_{n-2} \rangle$ -generic, each Y'_i is big. Applying the induction hypothesis to \mathcal{H}' and to $\langle Y'_0, \dots, Y'_{n-2} \rangle$, we obtain a function $\varrho' : (n - 1) \rightarrow A$ such that $\varrho'(i) \in Y'_i$ for all $i < n - 1$ and ϱ' isomorphically embeds \mathcal{H}' into \mathcal{G} . Let $\varrho = \varrho' \cup \{\langle n - 1, a \rangle\}$, that is, let ϱ be the extension of ϱ' that maps the last vertex of \mathcal{H} onto a . It is easy to see, that ϱ embeds \mathcal{H} into \mathcal{G} such that $\varrho(i) \in Y_i$ holds for all $i < n$. ■

Theorem 3.13 *For each finite graph \mathcal{H} there exists a constant $c(\mathcal{H}) \in \mathbb{R}^+$ with $c(\mathcal{H}) \leq 1$ such that for any finite graph \mathcal{G}^* the following holds: if \mathcal{G}^* has n vertices and does not contain complete and empty induced subgraphs of size $n^{c(\mathcal{H})}$ then \mathcal{H} can be isomorphically embedded into \mathcal{G}^* .*

Proof. Let \mathcal{H} be a finite graph and assume, seeking a contradiction, that for any $c \in \mathbb{R}^+$, $c \leq 1$ there exists a finite graph \mathcal{G}_c such that \mathcal{G}_c does not contain complete and empty induced subgraphs of size $|V(\mathcal{G}_c)|^c$, but \mathcal{H} cannot be isomorphically embedded into \mathcal{G}_c .

We claim, that $|V(\mathcal{G}_c)| \geq 2^{1/c}$. Indeed, because $c \leq 1$, any empty (respectively, complete) graph \mathcal{K} contains an empty (respectively, complete) subgraph of size $|V(\mathcal{K})|^c$, hence \mathcal{G}_c cannot be an empty or complete graph: it must contain an edge (that is, a two element complete induced subgraph) and a non-edge (that is, a two element empty induced subgraph). Therefore, $|V(\mathcal{G}_c)|^c \geq 2$, that is, $|V(\mathcal{G}_c)| \geq 2^{1/c}$, as desired.

By Theorem VI.3.12 of [16] there exists a regular ultrafilter $\mathcal{F} \subseteq \mathcal{P}(\aleph_1)$ with $\text{pcf}(\aleph_0, \mathcal{F}) = \aleph_1$. So, \mathcal{F} is \aleph_1 -regular: there exists $E \in [\mathcal{F}]^{\aleph_1}$ such that for any $i \in \aleph_1$ we have $\nu(i) := \{e \in E : i \in e\}$ is finite (we will assume, that $\aleph_1 \in E$, thus $\nu(i) \geq 1$, for all $i \in \aleph_1$). For each $i \in \aleph_1$ let $c(i) \in \mathbb{R}^+$ be such that $c(i) < 1/\nu(i)$. Since $1 \leq \nu(i)$ for all $i \in I$, it follows, that $\mathcal{G}_{c(i)}$ is defined for all $i \in I$. Finally, let

$$\mathcal{G} = \prod_{i \in \aleph_1} \mathcal{G}_{c(i)} / \mathcal{F}.$$

For every $i \in \aleph_1$ we have $c(i) < \frac{1}{\nu(i)}$; so $\nu(i)c(i) \leq 1$ whence $2^{\nu(i)c(i)} \leq 2$, that is, $2^{\nu(i)} < 2^{1/c(i)}$. Therefore, for any $i \in \aleph_1$ we have

$$|V(\mathcal{G}_{c(i)})| \geq 2^{1/c(i)} \geq 2^{\nu(i)}.$$

In addition, \mathcal{G} does not contain empty or complete big induced subgraphs, because of the following. Let $X = \langle X_i : i \in \aleph_1 \rangle / \mathcal{F} \subseteq V(\mathcal{G})$ be any decomposable, big set.

Then there exists $c \in \mathfrak{R}^+$ such that $I := \{i \in \aleph_1 : |X_i| \geq |V(\mathcal{G}_i)|^c\} \in \mathcal{F}$. Let $k \in \omega$ be such that $1/k < c$, and let J be the intersection of any k distinct elements of E . Then, for any $i \in I \cap J$ we have $\nu(i) \geq k$, hence

$$c(i) \leq \frac{1}{\nu(i)} \leq \frac{1}{k} \leq c,$$

so X_i does not induce a complete or empty subgraph of $\mathcal{G}_{c(i)}$. It follows, from the Loś Lemma, that X does not induce a complete or empty subgraph of \mathcal{G} .

According to the previous paragraph, the conditions of Lemma 3.9 are satisfied. Applying Lemma 3.9 (3) (with $X = V(\mathcal{G})$) we obtain a big $X' \subseteq V(\mathcal{G})$ such that if $A, B \subseteq X'$ are disjoint and big then there are $a, a' \in A$ and $b, b' \in B$ such that $\langle a, b \rangle \in E(\mathcal{G})$ and $\langle a', b' \rangle \notin E(\mathcal{G})$. Again by the previous paragraph, Theorem 3.12 can be applied: partitioning X' into $n := |V(\mathcal{H})|$ -many disjoint big sets Y_0, \dots, Y_{n-1} in an arbitrary way, there exist $a_0 \in Y_0, \dots, a_{n-1} \in Y_{n-1}$ such that the subgraph of \mathcal{G} induced by $\{a_0, \dots, a_{n-1}\}$ is isomorphic to \mathcal{H} . It is well known, that there exists a first order formula $\delta_{\mathcal{H}}$ (called the diagram of \mathcal{H}) such that for any graph \mathcal{M} we have $\mathcal{M} \models \delta_{\mathcal{H}}$ iff \mathcal{H} can be isomorphically embedded into \mathcal{M} ; for more details we refer to [1]. Hence - again by the Loś Lemma

$$\{i \in \aleph_1 : \mathcal{H} \text{ can be isomorphically embedded into } \mathcal{G}_{c(i)}\} \in \mathcal{F},$$

contradicting to the first sentence of the present proof. ■

4 Concluding Remarks

In this section we describe some problems, which remained open.

Open problem 4.1 *In Theorem 3.12, can the conditions on the ultrafilter \mathcal{F} be replaced by weaker ones such that Theorem 3.12 remains true ?*

The proof of Theorem 3.13 above establishes the existence of $c(\mathcal{H})$, but does not provide methods to compute, or estimate it from (the structure of) \mathcal{H} . Hence, the next problem is quite interesting, and remained completely open.

Open problem 4.2 *Develop methods estimating $c(\mathcal{H})$ from \mathcal{H} . In particular, is it true, that $c(\mathcal{H})$ may be chosen to be $2^{-|V(\mathcal{H})|}$?*

In the proof of Theorem 3.13 we found a single copy of \mathcal{H} in the ultraproduct graph \mathcal{G} , because for the present purposes this was enough. However, it is easy to see, that \mathcal{G} contains “many” isomorphic copies of each finite graph. In that direction the following problem remained open.

Open problem 4.3 Suppose \mathcal{H} is a finite graph and $\mathcal{G} = \prod_{i \in I} \mathcal{G}_i / \mathcal{F}$ is an ultraproduct of finite graphs such that if $X \subseteq V(\mathcal{G})$ is decomposable and induces a complete or empty subgraph of \mathcal{G} then X is small. Is it true, that the set of isomorphic copies of \mathcal{H} in \mathcal{G} is big in the following sense: there exists $c \in \mathbb{R}^+$ such that if $Y = \langle Y_i : i \in I \rangle / \mathcal{F} \subseteq {}^{V(\mathcal{H})}\mathcal{G}$ is a decomposable subset containing all the isomorphic copies of \mathcal{H} , then

$$\{i \in I : |Y_i| \geq |V(\mathcal{G}_i)|^{|V(\mathcal{H})|^c}\} \in \mathcal{F} ?$$

Is this true, if we assume further properties of the ultrafilter \mathcal{F} ?

In [3] it was shown, that the analogue of Theorem 3.13 for k -uniform hypergraphs is not true, if $k \geq 4$. This motivates our last problem.

Open problem 4.4 Can Theorem 3.13 be generalized to 3-uniform hypergraphs ?

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